A Unified Approach to the Purification of Nash Equilibria in Large Games

Guilherme Carmona

Universidade Nova de Lisboa

Faculdade de Economia

Campus de Campolide, 1099-032 Lisboa, Portugal

email: gcarmona@fe.unl.pt

telephone: (351) 21 380 1671

fax: (351) 21 388 6073

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Abstract

We present a purification result for incomplete information games with a large finite number of players that allows for compact metric spaces of actions and types. This result is then used to generalize the purification theorems of Schmeidler (1973), Rashid (1983) and Kalai (2004). Our proofs are elementary and rely on the Shapley-Folkman theorem.

1 Introduction

Mixed strategies are usually regarded as unappealing, either because they are hard to interpret or because they are considered as too complex for real players to use them. Motivated by this view, game-theorists have provided several purification theorems that describe when mixed strategies can be replaced by (approximately) equivalent pure strategies.

In this paper, we consider incomplete information games with a finite number of players in which each player's payoff depends only on his type-action character and on the average distribution of type-action characters of the others. Our purification result shows that, if players' types are independent and their payoff functions are selected from an equicontinuous family, then all (Bayesian) Nash equilibria of sufficiently large games can be approximately purified. More precisely, our result shows that for every Nash equilibrium of such games there exists a pure strategy approximate equilibrium that yields approximately the same payoff to all players.

More importantly, our result holds even when both the action space and the type space are compact metric spaces and not merely finite. This generality allow us to readily obtain the purification theorem of Rashid (1983) as a special case. In fact, Rashid considers the case of singleton type spaces (i.e., games with complete information) and finite action spaces.

Our result can also be used to generalize the purification theorems of Kalai (2004) and Schmeidler (1973), which is done by approximating the sort of games they consider by games in our class. In the case of the complete information games with a continuum of players considered by Schmeidler (1973), our exact purification result requires only that the action space is a countable, compact metric space, weakening in this way the finiteness assumption on the action space used by Schmeidler (1973). In the case of incomplete information games of Kalai (2004), our result dispenses with the finiteness of both the type and action spaces by assuming only that those spaces are compact and metric.

Our approach requires only elementary arguments. We start by establishing a purification result for games with a finite number of players, types and actions. The proof of this result is based on the Shapley-Folkman theorem (see Starr (1969)) in a similar way as was done by Rashid (1983). This basic result is then extended to the general case of games with compact metric spaces of types and actions.

The paper is organized as follows. In Section 2, we present our notation and definitions. Our purification result for games with a finite number of players, types and actions is presented in Section 3. In Section 4, this result is generalized to games with compact type and action spaces. Our generalizations of the purification theorems of Kalai (2004) and Schmeidler (1973) are in Section 5; in this section, we also discuss the work by Cartwright and

Wooders (2005) and those on games with diffused information. In Section 6, we present an example showing that it is not possible to approximately purify all Nash equilibria of sufficiently large games in which payoffs depend on the distribution of choices and not just on the average. Finally, some auxiliary results are presented in the Appendix.

2 Notation and Definitions

In the class of Bayesian games we consider, all players have a common type space T and a pure strategy space X. We assume that both T and X are compact metric spaces. Since the focus is on a property that depends on the number of players, we will index any game by the number of its players. Thus, G_n is a Bayesian game in which the set of players is $I_n = \{1, \ldots, n\}$, and each has T as his type space and X as his choice set.

Players are allowed to choose mixed strategies. In this context, a mixed strategy for a player is a function from types into a Borel probability measure on the set of his pure strategies. Let $\mathcal{M}(X)$ be the set of Borel probability measures on X endowed with the Prohorov metric ρ, which is known to metricize the weak convergence topology. Thus, a mixed strategy for player $i \in I_n$ is a Borel measurable function $\sigma_i : T \to \mathcal{M}(X)$. A strategy is pure if its values are degenerate probability measures on X. Thus, if for all $x \in X$, 1_x denotes the probability measure on X that attributes probability 1 to x, then a strategy $f_i : T \to \mathcal{M}(X)$ is pure if for all $t \in T$, there exists $x \in X$ such that $f_i(t) = 1_x$. Let Σ denote the set of all (mixed) strategies.

A game G_n is then specified by the vector of payoff functions, one for each player. We denote player i's payoff function by $V_i^n : T^n \times X^n \to \mathbb{R}$ for all $i \in I_n$ with the interpretation that the payoff of player i is $V_i^n(t_1, \dots, t_n, x_1 \dots, x_n)$ if players' types are $(t_1, \dots, t_n)$ and players' actions are $(x_1, \dots, x_n)$. Therefore, a game G_n is described by the vector $(V_1^n, \dots, V_n^n)$.

For all $i \in I_n$, player i's type is decided by nature according to $\tau_i \in \mathcal{M}(T)$. This probability measure, together with a strategy $\sigma_i : T \to \mathcal{M}(X)$, defines a measure $\breve{\sigma}_i$ on $T \times X$ as the unique measure satisfying

$$\breve{\sigma}_i(A \times B) = \int_A \sigma_i(B|t) \mathrm{d}\tau_i(t) \tag{1}$$

for all Borel measurable subsets A of T and B of X (see Ash (1972, Theorem 2.6.2, p. 97)).

In this paper we will focus on a special class of games in which each player's payoff depends on his type-action character and on the average distribution of type-action characters of the others. Given a strategy $\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma^n$ in a game G_n with n players, the average choice is:

$$\bar{\sigma} = \sum_{i \in I_n} \frac{\breve{\sigma}_i}{n}. \tag{2}$$

Similarly, for all $i \in I_n$, the average choice of the other players is

$$\bar{\sigma}_{-i} = \sum_{l \neq i} \frac{\breve{\sigma}_l}{n-1}. \tag{3}$$

Note that both $\bar{\sigma}$ and $\bar{\sigma}_{-i}$ are elements of $\mathcal{M}(T \times X)$.

Players' payoff functions are as follows. To each player i, we associate a continuous function $V_i^n : T \times X \times \mathcal{M}(T \times X) \to \mathbb{R}$ with the following interpretation: $V_i^n(t, x, \mu)$ is player i's payoff when he is of type t, plays action x and faces the distribution μ. Then, for all strategies σ, player i's payoff function is

$$U_i^n(\sigma) = \int_T \int_X V_i^n(t, x, \bar{\sigma}_{-i}) \mathrm{d}\sigma_i(x|t) \mathrm{d}\tau_i(t). \tag{4}$$

We denote this class of games by $\mathcal{H}$ and represent a game $G_n \in \mathcal{H}$ by $G_n = (I_n, V_n, \tau_n, T, X)$.

For convenience, let $U_i^n(\sigma_i(t), \sigma_{-i}|t) = \int_X V_i^n(t, x, \bar{\sigma}_{-i}) \mathrm{d}\sigma_i(x|t)$ be player i's payoff when his type is t, he plays $\sigma_i(t)$ and the other players play according to σ_{-i}. As a particular case, we have that $U_i^n(x, \sigma_{-i}|t) = V_i^n(t, x, \bar{\sigma}_{-i})$ if player i plays a pure strategy 1_x when his type is t. Using the above notation, we can write $U_i^n(\sigma) = \int_T U_i^n(\sigma_i(t), \sigma_{-i}|t) \mathrm{d}\tau_i(t)$.

We let $\mathcal{U}$ denote the space of all continuous, real-valued functions on $T \times X \times \mathcal{M}(T \times X)$ endowed with the sup norm. Thus, we can think of a game $G_n \in \mathcal{H}$ as a function V_n from I_n to $\mathcal{U}$ defined by $V_n(i) = V_i^n$ for all $i \in I_n$. Motivated by this remark, we let $V_n(I_n)$ denote the subset $\{V_1^n, \ldots, V_n^n\}$ of $\mathcal{U}$ consisting of all payoff functions in the game G_n.

For all $\varepsilon \geq 0$ and strategies $\sigma \in \Sigma^n$ let

$$\begin{aligned} E_n(\sigma, \varepsilon) = \{ & i \in I_n : V_i^n(t, x, \bar{\sigma}_{-i}) \geq V_i(t, \hat{x}, \bar{\sigma}_{-i}) - \varepsilon \\ & \text{for all } t \in T, x \in \operatorname{supp}(\sigma_i), \text{ and } \hat{x} \in X \} . \end{aligned} \tag{5}$$

The set $E_n(\sigma, \varepsilon)$ is the set of players who, for all $t \in T$, assign a strictly positive probability only to the actions that are within ε of their best response when all the others are playing according to σ_{-i}.

For all $\varepsilon \geq 0$ and $\eta \geq 0$, we say that σ is a *strong* (ε, η) *– equilibrium of a game* G_n if

$$\frac{|E_n(\sigma, \varepsilon)|}{n} \geq 1 - \eta. \tag{6}$$

Thus, in a strong (ε, η) – equilibrium a fraction of at least $1-\eta$ of the players assign a strictly positive probability only to the actions that are within ε of their best response. A strategy σ is a *Nash equilibrium of* G if σ is a strong (ε, η) – equilibrium of G_n for $\varepsilon = \eta = 0$.

We remark that this definition is equivalent to the more common one, according to which a strategy σ is a Nash equilibrium if for all $i \in I_n$ and $t \in T$, $\sigma_i(t)$ maximizes player i's payoff given that the others are using σ_{-i}.

Remark 1 *A strategy σ is a Nash equilibrium G_n if and only if*

$$U_i(\sigma_i(t), \sigma_{-i}|t) \geq \max_{x \in X} U_i(x, \sigma_{-i}|t)$$

for all $i \in I_n$ and $t \in T$.

Proof. Let $i \in I_n$ and $t \in T$. For convenience, let $\beta = \max_{x \in X} U_i(x, \sigma_{-i}|t)$. If σ is a Nash equilibrium, then $U_i(x, \sigma_{-i}|t) \geq \beta$ for all $x \in \text{supp}(\sigma_i(t))$. Hence,

$$U_i(\sigma_i(t), \sigma_{-i}|t) = \int_{\text{supp}(\sigma_i(t))} U_i(x, \sigma_{-i}|t) \mathrm{d}\sigma_i(x|t) \geq \beta.$$

Conversely, suppose that $U_i(\sigma_i(t), \sigma_{-i}|t) \geq \beta$. In order to reach a contradiction, assume that there is $\hat{x} \in \text{supp}(\sigma_i(t))$ such that $U_i(\hat{x}, \sigma_{-i}|t) < \beta$. Then, there exists $\varepsilon, \delta > 0$ such that $U_i(\tilde{x}, \sigma_{-i}|t) < \beta - \varepsilon$ for all $\tilde{x} \in B_\delta(\hat{x})$. Since $\hat{x} \in \text{supp}(\sigma_i(t))$, then $\sigma_i(B_\delta(\hat{x})|t) > 0$. Hence,

$$\begin{aligned} U_i(\sigma_i(t), \sigma_{-i}|t) &= \int_{X \setminus B_\delta(\hat{x})} U_i(x, \sigma_{-i}|t) \mathrm{d}\sigma_i(x|t) + \int_{B_\delta(\hat{x})} U_i(x, \sigma_{-i}|t) \mathrm{d}\sigma_i(x|t) \\ &\leq \beta - \varepsilon \sigma_i(B_\delta(\hat{x})|t) < \beta, \end{aligned} \tag{7}$$

a contradiction. ■

Let σ be a strong (ζ, η) – equilibrium of a game G_n. Then, we say that f is an ε – *purification of* σ if f is a pure strong $(\zeta + \varepsilon, \eta)$ – equilibrium and

$$|U_i^n(f) - U_i^n(\sigma)| < \zeta + \varepsilon, \tag{8}$$

for all $i \in E_n(\sigma, \zeta)$. Thus, under f, all the players that were only using ζ – best responses to σ_{-i} are playing actions that are $\zeta + \varepsilon$ – best responses to f_{-i}. Furthermore, their payoff under f is close to their payoff under σ.

Let K be a subset of $\mathcal{U}$. We say that K is *equicontinuous* (or, that the family K of functions is equicontinuous) if for all $\eta > 0$ there exists a $\delta > 0$ such that

$$|V(t, x, \mu) - V(s, y, \nu)| < \eta$$

whenever $\max\{d(t, s), d(x, y), \rho(\mu, \tau)\} < \delta$, $t, s \in T$, $x, y \in X$, $\mu, \tau \in \mathcal{M}(X)$ and $V \in K$ (see Rudin (1976, p. 156)). In our framework, equicontinuity can be interpreted as placing "a bound on the diversity of payoffs" (see Khan, Rath, and Sun (1997)).

3 Purification of Equilibria: Finite Type and Action Spaces

In this section we present a purification result for games with finite type and action spaces. It says that in all sufficiently large games, all strong approximate equilibria can be ε – purified, provided that players' payoff functions are selected from an equicontinuous family.

Lemma 1 *Let K be an equicontinuous subset of $\mathcal{U}$ and $m, p \in \mathbb{N}$. Then, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if:*

1. $n \geq N$*,*

2. $G_n \in \mathcal{H}$*,*

3. $V_n(I_n) \subseteq K$*,*

4. both T *and* X *are finite,* $|T| = p$ *and* $|X| = m$*,*

5. ν *is a strong* (ζ, η) *– equilibrium of* G_n *for some* $\zeta, \eta \geq 0$*,*

then there exists an ε *– purification* g *of* ν *such that* $\rho(\bar{\nu}, \bar{g}) < \varepsilon$ *and* $E_n(\nu, \zeta) \subseteq E_n(g, \zeta + \varepsilon)$*.*

Lemma 1 is our basic result, upon which all the others are based. Despite the fact that it implies the purification theorem of Rashid (1983), which corresponds to the case of $p = \zeta = \eta = 0$, its proof follows closely Rashid's arguments.[1] As there, the critical result for the construction of the purification strategies is the Shapley-Folkman Theorem.

Proof. Let $\varepsilon > 0$. Since K is equicontinuous, let $0 < \delta < \varepsilon$ be such that

$$|v(t, x, \tau) - v(s, y, \mu)| < \varepsilon/2 \tag{9}$$

whenever $\max\{d(t, s), d(x, y), \rho(\tau, \mu)\} < \delta$, $t, s \in T$, $x, y \in X$, $\tau, \mu \in \mathcal{M}(X)$ and $v \in K$.

Finally, let $N \in \mathbb{N}$ be such that $n > m$ and $4m^2p/n < \delta$ whenever $n \geq N$. In particular, $m^2p/n < \delta/2$ and $2/n < \delta/2$ for all $n \geq N$.

[1] Although the equicontinuity assumption was not made explicit in Rashid (1983), it is needed as we have shown in Carmona (2004b).

Let $n \geq N$ and G_n be a game in $\mathcal{H}$ such that $V_n(I_n) \subseteq K$, $|T| = p$ and $|X| = m$. Let ν be a strong (ζ, η) – equilibrium. Recall that

$$V_i(t, x_j, \bar{\nu}_{-i}) \geq V_i(t, x, \bar{\nu}_{-i}) - \zeta$$

for all $i \in E_n(\nu, \zeta)$, $t \in T$, $x \in X$ and $x_j \in \text{supp}(\nu_i(t))$.

Let $t \in T$ and define for all $i \in I_n$

$$S_{i,t} = \left\{ \frac{\tau_i(t)}{n} e_j : \nu_i(x_j|t) > 0 \right\}, \tag{10}$$

where $E = \{e_1, \ldots, e_m\}$ is the standard basis of $\mathbb{R}^m$. In particular, note that $S_{i,t} = \{0\}$ if $\tau_i(t) = 0$. However, if $\tau_i(t) > 0$, then $(n/\tau_i(t))S_{i,t} \subseteq E$.

Note that $(\tau_i(t)\nu_i(x_1|t)/n, \ldots, \tau_i(t)\nu_i(x_m|t)/n) \in \text{co}(S_{i,t})$ for all $i \in I_n$ and $t \in T$ since

$$\left(\frac{\tau_i(t)\nu_i(x_1|t)}{n}, \ldots, \frac{\tau_i(t)\nu_i(x_m|t)}{n} \right) = \sum_{j:\nu_i(x_j|t)>0} \nu_i(x_j|t) \frac{\tau_i(t)e_j}{n}, \tag{11}$$

$\nu_i(x_j|t) \geq 0$ for all j and $\sum_{j:\nu_i(x_j|t)>0} \nu_i(x_j|t) = 1$.

This implies that

$$(\bar{\nu}(x_1, t), \ldots, \bar{\nu}(x_m, t)) = \sum_{i=1}^{n} \frac{\tau_i(t)}{n} (\nu_i(x_1|t), \ldots, \nu_i(x_m|t))$$

is an element of

$$\text{co}\left(\sum_{i=1}^{n} S_{i,t} \right) = \sum_{i=1}^{n} \text{co}(S_{i,t}).$$

By the Shapley-Folkman Theorem (see Rashid (1983)), it follows that there are n points $\alpha_i(t) = (\alpha_i(x_1|t), \ldots, \alpha_i(x_m|t)) \in \text{co}(S_{i,t})$, with $i \in I_n$, such that

$$(\bar{\nu}(x_1, t), \ldots, \bar{\nu}(x_m, t)) = \sum_{i=1}^{n} (\alpha_i(x_1|t), \ldots, \alpha_i(x_m|t))$$

and

$$|\{i \in I_n : \alpha_i(t) \notin S_{i,t}\}| \leq m.$$

Since $S_{i,t} = \mathrm{co}(S_{i,t}) = \{0\}$ if $\tau_i(t) = 0$, then $\{i \in I_n : \alpha_i(t) \notin S_{i,t}\} \subseteq \{i \in I_n : \tau_i(t) > 0\}$. Hence,

$$|\{i \in I_n : \tau_i(t) > 0 \text{ and } n\alpha_i(t)/\tau_i(t) \notin E\}| \leq m. \tag{12}$$

Let $P_t = \{i \in I_n : \tau_i(t) > 0 \text{ and } n\alpha_i(t)/\tau_i(t) \in E\}$. Define a pure strategy g as follows: if $i \in P_t$, let e_j be such that $n\alpha_i(t)/\tau_i(t) = e_j$ and define $g_i(t) = 1_{x_j}$; if $i \in P_t^c$, choose $1 \leq j \leq m$ such that $\nu_i(x_j|t) > 0$ and define $g_i(t) = 1_{x_j}$. It then follows that

$$V_i(t, g_i(t), \bar{\nu}_{-i}) \geq V_i(t, x, \bar{\nu}_{-i}) - \zeta$$

for all $i \in E_n(\nu, \zeta)$, $t \in T$ and $x \in X$.

Note that, for all j and t,

$$\bar{g}(x_j, t) = \sum_{i \in P_t} \alpha_i(x_j|t) + \frac{1}{n} \sum_{i \in P_t^c} \tau_i(t) g_i(x_j|t). \tag{13}$$

Hence,

$$|\bar{\nu}(x_j, t) - \bar{g}(x_j, t)| = \frac{1}{n} \left| \sum_{i \in P_t^c : \tau_i(t) > 0} \tau_i(t) \left(\frac{n\alpha_i(x_j|t)}{\tau_i(t)} - g_i(x_j|t) \right) \right| \leq \frac{m}{n}. \tag{14}$$

By Lemma 12, it follows that

$$\rho(\bar{\nu}, \bar{g}) \leq m^2 p/n < \delta/2 < \varepsilon.$$

Also by Lemma 13, it follows that

$$\begin{aligned} \rho(\bar{\nu}_{-i}, \bar{g}_{-i}) &\leq \rho(\bar{\nu}_{-i}, \bar{\nu}) + \rho(\bar{\nu}, \bar{g}) + \rho(\bar{g}, \bar{g}_{-i}) \\ &\leq \frac{1}{n} + \frac{\delta}{2} + \frac{1}{n} < \delta, \end{aligned} \tag{15}$$

for all $i \in I_n$.

Hence, for all $i \in E_n(\nu, \zeta)$, $t \in T$ and $x \in X$, we obtain

$$\begin{aligned} V_i(t, g_i(t), \bar{g}_{-i}) &> V_i(t, g_i(t), \bar{\nu}_{-i}) - \frac{\varepsilon}{2} \\ &\geq V_i(t, x, \bar{\nu}_{-i}) - \frac{\varepsilon}{2} - \zeta \\ &> V_i(t, x, \bar{g}_{-i}) - \varepsilon - \zeta. \end{aligned} \tag{16}$$

Therefore, $E_n(\nu, \zeta) \subseteq E_n(g, \zeta + \varepsilon)$. Since ν is a (ζ, η) – equilibrium, then $|E_n(\nu, \zeta)|/n \geq 1 - \eta$ and so g is a pure $(\zeta + \varepsilon, \eta)$ – equilibrium of G_n.

Since ν is a strong (ζ, η) – equilibrium, it follows that $|V_i(t, x, \bar{\nu}_{-i}) - V_i(t, \hat{x}, \bar{\nu}_{-i})| \leq \zeta$ for all $i \in E_n(\nu, \zeta)$, $t \in T$ and $x, \hat{x} \in \operatorname{supp}(\nu_i(t))$. Hence,

$$\begin{aligned} &\left| \sum_{j=1}^{m} \nu_i(x_j|t) V_i(t, x_j, \bar{\nu}_{-i}) - V_i(t, g_i(t), \bar{g}_{-i}) \right| \leq \\ &\left| \sum_{j=1}^{m} \nu_i(x_j|t) V_i(t, x_j, \bar{\nu}_{-i}) - V_i(t, g_i(t), \bar{\nu}_{-i}) \right| + \\ &|V_i(t, g_i(t), \bar{\nu}_{-i}) - V_i(t, g_i(t), \bar{g}_{-i})| < \zeta + \varepsilon. \end{aligned} \tag{17}$$

Thus,

$$|U_i(\nu) - U_i(g)| < \zeta + \varepsilon$$

for all $i \in E_n(\nu, \zeta)$ and so g is an ε – purification of ν. ■

4 Purification of Equilibria: Compact Type and Action Spaces

In order to prove Lemma 1, it is crucial that both the type and action spaces be finite. In this case, probability measures on their product $T \times X$ can be regarded as points in the standard unit simplex of some finite-dimensional euclidian space, which makes it possible to use the Shapley-Folkman Theorem.

This tool can no longer be used when the type and action spaces are just compact metric spaces. Nevertheless, as our main result shows, the conclusion of Lemma 1 still holds: all strong approximate equilibria can be ε – purified in sufficiently large equicontinuous games.

Theorem 1 *Let K be an equicontinuous subset of $\mathcal{U}$. Then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with the following property:*

If $n \geq N$ and $G_n = (I_n, V_n, \tau_n, T, X) \in \mathcal{H}$ is such that $V_n(I_n)$ is a subset of K, then for all $\zeta, \eta \geq 0$ and all strong (ζ, η) – equilibrium σ of G_n, there exists an ε – purification f of σ such that $\rho(\bar{\sigma}, \bar{f}) < \varepsilon$.

A standard purification result for Nash equilibria is simply obtained by considering the particular case of $\zeta = \eta = 0$.

Corollary 1 *Let K be an equicontinuous subset of $\mathcal{U}$. Then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with the following property:*

If $n \geq N$ and $G_n = (I_n, V_n, \tau_n, T, X) \in \mathcal{H}$ is such that $V_n(I_n)$ is a subset of K, then all Nash equilibria of G_n can be ε – purified.

The method of proof for Theorem 1 consists in approximating games with compact type and action spaces by games in which these spaces are finite. In order for this approximation to be useful, it must be possible to map strategies in one type of game into strategies in the other in such a way that they preserve their equilibrium properties.

As our main approximation lemma (Lemma 4) shows, a sufficient condition for the existence of such mapping is that strategies induce distributions on $T \times X$ that are close in both the finite and the compact cases. The following two lemmas show how this property can be achieved by making sufficiently fine partitions of T and X.

The notion of a partition that we use is as follows. Let Y be a metric space and $\gamma > 0$. A γ *– pointed partition of* Y is a finite measurable partition $\{Y_1, \ldots, Y_m\}$ of Y and corresponding points $\{y_1, \ldots, y_m\}$ such that for all $j \in \{1, \ldots, k\}$, $y_j \in Y_j$ and $Y_j \subseteq B_\gamma(y_j)$.

Since X is a compact metric space, then, for all $\gamma > 0$, there is a γ – pointed partition of X. In fact, let $\{\hat{x}_1, \ldots, \hat{x}_m\}$ be such that $X = \cup_{j=1}^m B_{\gamma/2}(\hat{x}_j)$ and define $X_j = B_{\gamma/2}(\hat{x}_j) \setminus \left(\cup_{l=1}^{j-1} B_\gamma(\hat{x}_l)\right)$ for all $j = 1, \ldots, m$. The partition is $\{X_j : X_j \neq \emptyset\}$. For all such X_j, let $x_j \in X_j$; it is easy to see that $X_j \subseteq B_\gamma(x_j)$. Hence, the γ – pointed partition is $\{(X_j, x_j) : X_j \neq \emptyset\}$. By the same argument, there is a γ – partition of T for all $\gamma > 0$.

The following lemma shows that if two probability measures coincide in all the measurable rectangles $T_l \times X_j$ induced by a γ – pointed partition, then their distance is no greater than 2γ.

Lemma 2 *Let $\{X_1, \ldots, X_m\}$ and $\{T_1, \ldots, T_p\}$ be γ – partitions of X and T, respectively, for some $\gamma > 0$. Let μ and ν be probability measures in $T \times X$.*

If $\mu(T_l \times X_j) = \nu(T_l \times X_j)$ for all $l = 1, \ldots, p$ and $j = 1, \ldots, m$, then

$$\rho(\mu, \nu) \leq 2\gamma.$$

Proof. Let $D \subseteq T \times X$ be Borel measurable. Let $A = \{(l, j) : \mu((T_l \times X_j) \cap D) > 0\}$ and $B = \{(l, j) : \nu((T_l \times X_j) \cap \overline{B}_{2\gamma}(D)) > 0\}$. Note that if $(l,j) \in A$, then $(T_l \times X_j) \cap D \neq \emptyset$, which implies that $(t_l, x_j) \in \overline{B}_\gamma(D)$. Hence, $(t,x) \in \overline{B}_{2\gamma}(D)$ for all $t \in T_l$ and $x \in X_j$, i.e., $T_l \times X_j \subseteq \overline{B}_{2\gamma}(D)$.

Also, it follows from $\mu((T_l \times X_j) \cap D) > 0$ that $\nu(T_l \times X_j) = \mu(T_l \times X_j) > 0$ and so $\nu((T_l \times X_j) \cap \overline{B}_{2\gamma}(D)) = \nu(T_l \times X_j) > 0$. Thus, $A \subseteq B$.

Therefore,

$$\begin{aligned} \mu(D) &= \sum_{(l,j)\in A} \mu((T_l \times X_j) \cap D) \leq \sum_{(l,j)\in A} \mu(T_l \times X_j) \\ &= \sum_{(l,j)\in A} \nu(T_l \times X_j) = \sum_{(l,j)\in A} \nu((T_l \times X_j) \cap \overline{B}_{2\gamma}(D)) \\ &\leq \sum_{(l,j)\in B} \nu((T_l \times X_j) \cap \overline{B}_{2\gamma}(D)) = \nu(\overline{B}_{2\gamma}(D)) < \nu(\overline{B}_{2\gamma}(D)) + 2\gamma. \end{aligned} \tag{18}$$

By Fristedt and Gray (1997, Problem 29, p. 364), we also have that $\nu(D) \leq \mu(\overline{B}_{2\gamma}(D)) + 2\gamma$ and so $\rho(\mu, \nu) \leq 2\gamma$. ∎

It follows from Lemma 2 that we can control the distance of probability measures using γ – pointed partitions provided that they coincide on the rectangles induced by it. Hence, if we want to replace a strategy by another one in a way that the distributions induced by the two are close, then that property is certainly useful.

However, we also want to replace a strategy φ_i in a game with compact type and action spaces with another strategy in a game with finite type and action spaces. In order to achieve these properties, it is useful that φ_i be constant in each cell of the γ – pointed partition of T and that the support of $\varphi_i(t)$ be contained in a finite subset of X for all $t \in T$.

The two properties above are formalized in the following notion. Let $\{(X_1, x_1), \ldots, (X_m, x_m)\}$ and $\{(T_1, t_1), \ldots, (T_p, t_p)\}$ be γ – partitions of X and T, respectively, for some $\gamma > 0$. A strategy φ_i is *subordinate to* $\{(X_1, x_1),$

$\ldots,(X_m,x_m)\}$ *and* $\{(T_1,t_1),\ldots,(T_p,t_p)\}$ if $\varphi_i(t)=\varphi_i(t_l)$ for all $t\in T_l$ and $l=1,\ldots,p$ and $\mathrm{supp}(\varphi_i(t))\subseteq\{x_1,\ldots,x_m\}$ for all $t\in T$.

The following lemma shows that any strategy can be replaced with another one in such a way that all the desired properties we have discussed are satisfied.

Lemma 3 *Let* $\{X_1,\ldots,X_m\}$ *and* $\{T_1,\ldots,T_p\}$ *be* γ *– partitions of* X *and* T, *respectively, for some* $\gamma>0$ *and let* $G\in\mathcal{H}$ *be a game.*

Then, for all strategies σ *of* G, *there exists a strategy* φ *such that for all* $i\in I$,

1. φ_i *is subordinate to* $\{X_1,\ldots,X_m\}$ *and* $\{T_1,\ldots,T_m\}$ *and*

2. $\breve{\varphi}_i(T_l\times X_j)=\breve{\sigma}_i(T_l\times X_j)$ *for all* $l=1,\ldots,p$ *and* $j=1,\ldots,m$.

Proof. Let $i\in I$, $l=1\ldots p$ and $j=1,\ldots m$. Define

$$\varphi_i(x_j|t_l)=\frac{\int_{T_l}\sigma_i(X_j|t)\mathrm{d}\tau_i(t)}{\tau_i(T_l)} \tag{19}$$

if $\tau_i(T_l)>0$,

$$\varphi_i(x_j|t_l)=\sigma_i(X_j|t_l) \tag{20}$$

if $\tau_i(T_l)=0$ and

$$\varphi_i(t)=\varphi_i(t_l) \tag{21}$$

for all $t\in T_l$. Clearly, φ_i is subordinate to $\{(X_1,x_1),\ldots,(X_m,x_m)\}$ and $\{(T_1,t_1),\ldots,(T_m,t_m)\}$.

Since $\breve{\sigma}_i(T_l\times X_j)=\int_{T_l}\sigma_i(X_j|t)\mathrm{d}\tau_i(t)$ and $\breve{\varphi}_i(T_l\times X_j)=\tau_i(T_l)\varphi_i(x_j|t_l)$, it follows that

$$\breve{\sigma}_i(T_l\times X_j)=\breve{\varphi}_i(T_l\times X_j).$$

■

The following is our approximation lemma. It shows that we can associate to any game with compact type and action spaces a game with finite type and action spaces in such a way that approximate equilibria in one are also approximate equilibria in the other (with possibly different approximation levels).

Lemma 4 *Let K be an equicontinuous subset of $\mathcal{U}$ and $\alpha > 0$. Then, there exist $m, p \in \mathbb{N}$ such that the following holds:*

For all games $G_n = (I_n, V_n, \tau, T, X) \in \mathcal{H}$ satisfying $V_n(I_n) \subseteq K$, there exists a game $\hat{G}_n = (I_n, \hat{V}_n, \hat{\tau}, \hat{T}, \hat{X}) \in \mathcal{H}$ such that

1. *both $\hat{T}$ and $\hat{X}$ are finite, $|\hat{T}| = p$ and $|\hat{X}| = m$;*
2. *$\hat{V}_i^n = V_i^n|_{\hat{T} \times \hat{X} \times \mathcal{M}(\hat{T} \times \hat{X})}$ for all $i \in I_n$;*
3. *If σ is a strong (ζ, η) – equilibrium of G_n for some $\zeta, \eta \geq 0$, then there exists a strong $(\zeta + \alpha, \eta)$ – equilibrium ν of $\hat{G}_n$ such that $\rho(\bar{\sigma}, \bar{\nu}) < \alpha$, $E_n(\sigma, \zeta) \subseteq E_n(\nu, \zeta + \alpha)$ and $|U_i^n(\sigma) - U_i^n(\nu)| < \alpha$ for all $i \in I_n$; and*
4. *If g is a pure strategy strong (ζ, ν) – equilibrium of $\hat{G}_n$ for some $\zeta, \eta \geq 0$, then there exists a pure strategy strong $(\zeta + \alpha, \eta)$ – equilibrium f of G_n such that $\rho(\bar{f}, \bar{g}) < \alpha$, $E_n(g, \zeta) \subseteq E_n(f, \zeta + \alpha)$ and $|U_i^n(f) - U_i^n(g)| < \alpha$ for all $i \in I_n$.*

Proof. Let $\alpha > 0$ and let $0 < \gamma < \alpha/4$. Since K is equicontinuous, there exists $0 < \delta < \alpha$ such that $\max\{d(t, s), d(x, y), \rho(\mu, \nu)\} < \delta$ implies that $|v(t, x, \mu) - v(s, y, \nu)| < \gamma$ for all $t, s \in T$, $x, y \in X$, $\mu, \nu \in \mathcal{M}(T \times X)$ and $v \in K$.

Since X and T are compact metric spaces, there exist $\delta/4$ – partitions $\{(X_1, x_1), \ldots, (X_m, x_m)\}$ and $\{(T_1, t_1), \ldots, (T_p, t_p)\}$ of X and T, respectively.

Define $\hat{X} = \{x_1, \ldots, x_m\}$, $\hat{T} = \{t_1, \ldots, t_p\}$ and $\hat{\tau}$ by $\hat{\tau}_i(t_l) = \tau_i(T_l)$ for all $i \in I_n$ and $l = 1, \ldots, p$.

Let $G_n = (I_n, V_n, \tau_n, T, X)$ be such that $V_n(T_n)$ is a finite subset of K. Define $\hat{V}_n^i = V_n^i|_{\hat{T}\times\hat{X}\times\mathcal{M}(\hat{T}\times\hat{X})}$ for all $i \in I_n$ and $\hat{G}_n = (I_n, \hat{V}_n, \hat{\tau}, \hat{T}, \hat{X})$. Clearly, properties 1 and 2 in the statement of the Lemma are satisfied.

Let σ be a strong (ζ, η) – equilibrium of G_n for some $\zeta, \eta \geq 0$. By Lemma 3, there exists a strategy φ such that for all $i \in I$,

1. φ_i is subordinate to $\{(X_1, x_1), \ldots, (X_m, x_m)\}$ and $\{(T_1, t_1), \ldots, (T_m, t_m)\}$ and
2. $\breve{\varphi}_i(T_l \times X_j) = \breve{\sigma}_i(T_l \times X_j)$ for all $l = 1, \ldots, p$ and $j = 1, \ldots, m$.

Thus, by Lemma 2, $\rho(\varphi_i, \sigma_i) \leq \delta/2$ for all $i \in I_n$, which implies that $\bar{\varphi}_{-i}$ is close to $\bar{\sigma}_{-i}$. In fact, for all $i \in I_n$,

$$\rho(\bar{\varphi}_{-i}, \bar{\sigma}_{-i}) \leq \delta/2 \tag{22}$$

since, if $D \subseteq T \times X$ is Borel measurable and $k \neq i$, then

$$\bar{\sigma}_{-i}(D) = \frac{\sum_{k\neq i} \sigma_k(D)}{n-1} \leq \frac{\sum_{k\neq i} \left[\varphi_k(\overline{B}_{\delta/2}(D)) + \delta/2\right]}{n-1} = \bar{\varphi}_{-i}(\overline{B}_{\delta/2}(D)) + \frac{\delta}{2}. \tag{23}$$

By Fristedt and Gray (1997, Problem 29, p. 364), we also have that $\bar{\varphi}_{-i}(D) \leq \bar{\sigma}_{-i}(\overline{B}_{\delta/2}(D)) + \delta/2$ and so the claim follows. Similarly, $\rho(\bar{\sigma}, \bar{\varphi}) \leq \delta/2$.

Note that $\rho(\bar{\varphi}_{-i}, \bar{\sigma}_{-i}) \leq \delta/2$ implies that

$$|V_i(t, x, \bar{\varphi}_{-i}) - V_i(t, x, \bar{\sigma}_{-i})| < \gamma$$

for all $i \in I_n$, $t \in T$ and $x \in X$.

We next show that the payoff under σ is close to that under φ.

Claim 1 *For all $i \in I_n$, $|U_i(\sigma) - U_i(\varphi)| < 3\gamma$.*

Proof. Let $1 \leq j \leq m$ and $1 \leq l \leq p$. Then,

$$\begin{aligned}
&\left| \int_{T_l} \int_{X_j} V_i(t, x, \bar{\sigma}_{-i}) \mathrm{d}\sigma_i(x|t)\mathrm{d}\tau_i - \breve{\sigma}_i(T_l \times X_j) V_i(t_l, x_j, \bar{\sigma}_{-i}) \right| \\
&= \left| \int_{T_l} \int_{X_j} \left[V_i(t, x, \bar{\sigma}_{-i}) - V_i(t_l, x_j, \bar{\sigma}_{-i}) \right] \mathrm{d}\sigma_i(x|t)\mathrm{d}\tau_i \right| \\
&< \breve{\sigma}_i(T_l \times X_j)\gamma.
\end{aligned} \tag{24}$$

Likewise,

$$\left| \int_{T_l} \int_{X_j} V_i(t, x, \bar{\varphi}_{-i}) \mathrm{d}\varphi_i(x|t)\mathrm{d}\tau_i - \breve{\varphi}_i(T_l \times X_j) V_i(t_l, x_j, \bar{\varphi}_{-i}) \right| < \breve{\varphi}_i(T_l \times X_j)\gamma. \tag{25}$$

Since $\breve{\sigma}_i(T_l \times X_j) = \breve{\varphi}_i(T_l \times X_j)$, it follows that

$$\begin{aligned}
&\left| \int_{T_l} \int_{X_j} V_i(t, x, \bar{\sigma}_{-i}) \mathrm{d}\sigma_i(x|t)\mathrm{d}\tau_i - \int_{T_l} \int_{X_j} V_i(t, x, \bar{\varphi}_{-i}) \mathrm{d}\varphi_i(x|t)\mathrm{d}\tau_i \right| \\
&< 2\gamma\breve{\sigma}_i(T_l \times X_j) + \breve{\sigma}_i(T_l \times X_j) \left| V_i(t_l, x_j, \bar{\sigma}_{-i}) - V_i(t_l, x_j, \bar{\varphi}_{-i}) \right| \\
&< 3\gamma\breve{\sigma}_i(T_l \times X_j).
\end{aligned} \tag{26}$$

Hence,

$$|U_i(\sigma) - U_i(\varphi)| < \sum_{l,j} 3\gamma\breve{\sigma}_i(T_l \times X_j) = 3\gamma. \tag{27}$$

■

We next show that φ is a strong $(\zeta+2\gamma,\eta)$ – equilibrium. Let $i \in E_n(\sigma,\zeta)$, $t \in T$, $x \in X$ and $x_j \in \mathrm{supp}(\varphi_i(t))$. Then,

$$\begin{aligned} V_i(t, x_j, \bar{\varphi}_{-i}) &> V_i(t, x_j, \bar{\sigma}_{-i}) - \gamma \\ &\geq V_i(t, x, \bar{\sigma}_{-i}) - \gamma - \zeta \\ &> V_i(t, x, \bar{\varphi}_{-i}) - 2\gamma - \zeta. \end{aligned} \tag{28}$$

Thus, $E_n(\sigma,\zeta) \subseteq E_n(\varphi,\zeta+2\gamma)$ and so φ is a strong $(\zeta+2\gamma,\eta)$ – equilibrium, as claimed.

Define

$$\nu_i(x_j|t_l) = \varphi_i(x_j|t_l) \tag{29}$$

for all $i \in I_n$, $l = 1, \ldots, p$ and $j = 1, \ldots, m$ and so

$$\breve{\nu}_i(\{(t_l, x_j)\}) = \hat{\tau}_i(t_l)\nu_i(x_j|t_l).$$

Although $\breve{\nu}_i$ is a measure on $\hat{T} \times \hat{X}$, it can be extended in an obvious way to a measure on $T \times X$ by defining

$$\breve{\nu}_i(D) = \sum_{(l,j):(t_l,x_j)\in D} \breve{\nu}_i(\{(t_l, x_j)\}). \tag{30}$$

for all Borel measurable subsets D of $T \times X$. Hence, we can think of $\breve{\nu}_i$ as a measure on $T \times X$.

With the above remark in mind, note that $\breve{\nu}_i(T_l \times X_j) = \tau(T_l)\varphi_i(x_j|t_l) = \breve{\varphi}_i(T_l \times X_j)$. As in inequality (22) above, this implies that $\rho(\bar{\nu}_{-i}, \bar{\varphi}_{-i}) \leq \delta/2$ for all $i \in I_n$ and $\rho(\bar{\nu}, \bar{\varphi}) \leq \delta/2$. Since φ is a strong $(\zeta + 2\gamma, \eta)$ – equilibrium of G_n, then $E_n(\varphi, \zeta + 2\gamma) \subseteq E_n(\nu, \zeta + 4\gamma)$ and ν is a strong $(\zeta + 4\gamma, \eta)$ – equilibrium of $\hat{G}_n$. Furthermore, $E_n(\sigma,\zeta) \subseteq E_n(\nu, \zeta+\alpha)$ (since $E_n(\nu, \zeta + 4\alpha) \subseteq E_n(\nu, \zeta + \alpha)$) and $\rho(\bar{\sigma}, \bar{\nu}) \leq \delta < \alpha$.

Moreover, since

$$U_i(\varphi) = \sum_{l=1}^{m} \int_{T_l} \left[\sum_{j=1}^{m} V_i(t, x_j, \bar{\varphi}_{-i}) \varphi_i(x_j|t_l) \right] \mathrm{d}\tau_i(t), \tag{31}$$

$$\begin{aligned} U_i(\nu) &= \sum_{l=1}^{m} \hat{\tau}_i(t_l) \sum_{j=1}^{m} V_i(t_l, x_j, \bar{\nu}_{-i}) \nu_i(x_j|t_l) \\ &= \sum_{l=1}^{m} \int_{T_l} \left[\sum_{j=1}^{m} V_i(t_l, x_j, \bar{\nu}_{-i}) \nu_i(x_j|t_l) \right] \mathrm{d}\tau_i(t), \end{aligned} \tag{32}$$

and $\nu_i(x_j|t_l) = \varphi_i(x_j|t_l)$ for all $l = 1, \ldots, p$ and $j = 1, \ldots, m$, it follows that

$$\begin{aligned} &|U_i(\varphi) - U_i(\nu)| \leq \\ &\sum_{l=1}^{p} \int_{T_l} \left[\sum_{j=1}^{m} |V_i(t, x_j, \bar{\varphi}_{-i}) - V_i(t_l, x_j, \bar{\nu}_{-i})| \, \varphi_i(x_j|t_l) \right] \mathrm{d}\tau_i(t) < \gamma, \end{aligned} \tag{33}$$

for all $i \in I_n$. Therefore, inequality (33) and Claim 1, imply that

$$|U_i(\sigma) - U_i(\nu)| \leq |U_i(\sigma) - U_i(\varphi)| + |U_i(\varphi) - U_i(\nu)| < 4\gamma < \alpha$$

for all $i \in I_n$, which proves property 3.

Finally, we prove property 4. Let g be a pure strategy strong (ζ, η) – equilibrium of $\hat{G}_n$. Define

$$f_i(t) = g_i(t_l) \tag{34}$$

if $t \in T_l$ for all $i \in I_n$. Then, for all $i \in I_n$, $l = 1, \ldots, p$ and $j = 1, \ldots, m$, $\breve{f}_i(T_l \times X_j) = \tau_i(T_l) = \breve{g}_i(T_l \times X_j)$ if $f_i(t_l) = 1_{x_j}$ and $\breve{f}_i(T_l \times X_j) = 0 = \breve{g}_i(T_l \times X_j)$ if $f_i(t_l) \neq 1_{x_j}$. Thus, $\rho(\bar{f}_{-i}, \bar{g}_{-i}) \leq \delta/2$ for all $i \in I_n$ and also, $\rho(\bar{f}, \bar{g}) \leq \delta/2 < \alpha$.

Therefore, f is a strong $(\zeta + 2\gamma, \eta)$ – equilibrium of G_n since for all $i \in E_n(g, \zeta)$, $t \in T$ and $x \in X$, if $t \in T_l$ and $x \in X_j$, then

$$\begin{aligned} V_i(t, f_i(t), \bar{f}_{-i}) &> V_i(t_l, g_i(t_l), \bar{g}_{-i}) - \gamma \\ &\geq V_i(t_l, x_j, \bar{g}_{-i}) - \zeta - \gamma \\ &> V_i(t, x, \bar{f}_{-i}) - \zeta - 2\gamma. \end{aligned} \tag{35}$$

Thus, $E_n(g, \zeta) \subseteq E_n(f, \zeta + \alpha)$. Furthermore, arguing as in (33), it follows that $|U_i(g) - U_i(f)| < \gamma$ for all $i \in I_n$. Since $2\gamma < \alpha$, property 4 follows. ■

Combining Lemma 4 with Lemma 1, it is easy to prove our purification result.

Proof of Theorem 1. Let $\varepsilon > 0$ and $0 < \alpha < \varepsilon/4$. Let $m, p \in \mathbb{N}$ be given by Lemma 4 and corresponding to K and α. Let $N \in \mathbb{N}$ be given by the Lemma 1 and corresponding to K, α, m and p.

Let $n \geq N$ and $G_n \in \mathcal{H}$ be such that $V_n(I_n) \subseteq K$. Let σ be a strong (ζ, η) – equilibrium of G_n for some $\zeta, \eta \geq 0$. By Lemma 4, let $\hat{G}_n \in \mathcal{H}$ be such that $|\hat{T}| = p$, $|\hat{X}| = m$ and ν be a strong $(\zeta + \alpha, \eta)$ – equilibrium of $\hat{G}_n$ satisfying $\rho(\bar{\sigma}, \bar{\nu}) < \alpha$, $E_n(\sigma, \zeta) \subseteq E_n(\nu, \zeta + \alpha)$ and $|U_i(\sigma) - U_i(\nu)| < \alpha$ for all $i \in I_n$.

By Lemma 1, there exists a pure strategy strong $(\zeta+2\alpha, \eta)$ – equilibrium g of $\hat{G}_n$ such that $\rho(\bar{\nu}, \bar{g}) < \alpha$ and $|U_i(\nu) - U_i(g)| < \zeta + 2\alpha$ for all $i \in E_n(\nu, \zeta+\alpha)$. Since, $E_n(\sigma, \zeta) \subseteq E_n(\nu, \zeta + \alpha)$, it follows that $|U_i(\nu) - U_i(g)| < \zeta + 2\alpha$ for all $i \in E_n(\sigma, \zeta)$.

By Lemma 4 again, let f be a pure strategy strong $(\zeta + 3\alpha)$ – equilibrium of G_n satisfying $\rho(\bar{g}, \bar{f}) < \alpha$ and $|U_i(g) - U_i(f)| < \alpha$ for all $i \in I_n$.

Since $4\eta < \varepsilon$, it follows that f is a strong $(\zeta + \varepsilon, \eta)$ – equilibrium of G_n, $\rho(\bar{\sigma}, \bar{f}) < \varepsilon$ and $|U_i(f) - U_i(\sigma)| < 4\alpha < \varepsilon$ for all $i \in E_n(\sigma, \zeta)$. Thus, f is an

ε – purification of σ. ■

5 Related Literature

In this section we discuss the work of Cartwright and Wooders (2005) and the purification results for games with diffused information. More importantly, we extend the purification theorems of Kalai (2004) and Schmeidler (1973).

The generalization of Kalai's purification theorem is based on a generalized version of Theorem 1, which is accomplished by the following generalization of our framework.

A class of games $\mathcal{G}$ is defined by a sequence of functional spaces $\{\mathcal{F}_n\}_{n=1}^{\infty}$ and a sequence of functions $\{F_n\}_{n=1}^{\infty}$ satisfying the following properties for all $n \in \mathbb{N}$:

1. if $G_n \in \mathcal{G}$, then $V_i^n \in \mathcal{F}_n$ and

2. F_n maps $(\tau_1, \dots, \tau_n)$, $(V_1, \dots, V_n)$ and $(\sigma_1, \dots, \sigma_n)$ into $\mathbb{R}^n$ with the interpretation that player i's payoff $U_i^n(\sigma)$ resulting from a strategy $\sigma = (\sigma_1, \dots, \sigma_n)$ is equal to $F_i^n(\tau_1, \dots, \tau_n, V_1, \dots, V_n, \sigma_1, \dots, \sigma_n)$ for all $i \in I_n$.

In the particular case of the class $\mathcal{H}$, for all $n \in \mathbb{N}$, the functional space is $\mathcal{U}$ and $F_n(\tau_1, \dots, \tau_n, V_1, \dots, V_n, \sigma_1, \dots, \sigma_n) = (u_1, \dots, u_n)$ if for all $i \in I_n$

$$u_i = \int_T \int_X V_i(t, x, \bar{\sigma}_{-i}) \mathrm{d}\sigma_i(x|t) \mathrm{d}\tau_i(t).$$

Let $\mathcal{G} = \{\mathcal{F}_n, F_n\}_{n=1}^{\infty}$ be a class of games, $C_n \subseteq \mathcal{F}_n$ for all $n \in \mathbb{N}$ and $K \subseteq \mathcal{U}$. Then $(\mathcal{G}, \{C_n\}_{n=1}^{\infty})$ *can be approximated by* $(\mathcal{H}, K)$ if the following property holds: For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, $G_n = (I_n, V_n, \tau_n, T, X)$ belongs to $\mathcal{G}$ and $V_n(I_n) \subseteq C_n$, then there exists $\hat{G}_n = (I_n, \hat{V}_n, \tau_n, T, X) \in \mathcal{H}$ satisfying $\hat{V}_n(I_n) \subseteq K$ and

$$|U_i^n(\sigma) - \hat{U}_i^n(\sigma)| < \varepsilon \tag{36}$$

for all $t \in T_n$ and all strategies $\sigma \in \Sigma^n$. The following corollary of Theorem 1 is an obvious, but useful, consequence of the above definition.

Corollary 2 *Let K be an equicontinuous subset of $\mathcal{U}$ and assume that ($\mathcal{G}$, $\{C_n\}_{n=1}^{\infty}$) can be approximated by $(\mathcal{H}, K)$. Then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with the following property:*

If $n \geq N$ and $G_n = (I_n, V_n, \tau_n, T, X) \in \mathcal{G}$ is such that $V_n(I_n)$ is a subset of C_n, then for all $\zeta, \eta \geq 0$ and all strong (ζ, η) – equilibrium σ of G_n, there exists an ε – purification f of σ such that $\rho(\bar{\sigma}, \bar{f}) < \varepsilon$.

5.1 Kalai (2004)

In order to generalize Kalai's purification theorem, we start by defining the class of games he considers, which we denote by $\mathcal{K}$. In this class, the functional space is $\mathcal{U}$ for all $n \in \mathbb{N}$. We next define F_n.

Let $C = T \times X$, $i \in I_n$ and $c_{-i} \in C_{-i} = C^{n-1}$. Define a measure $\mathrm{emp}_{c_{-i}} \in \mathcal{M}(T \times X)$ by

$$\mathrm{emp}_{c_{-i}}(B) = \frac{|\{j \neq i : c_j \in B\}|}{n-1} \tag{37}$$

for all Borel measurable subsets B of $T \times X$. The quantity $\mathrm{emp}_{c_{-i}}(B)$ is simply the fraction of players other than player i with type-action characters in the set B.

The following lemma shows that if two vectors c_{-i} and $\hat{c}_{-i}$ are close, then so will be $\mathrm{emp}_{c_{-i}}$ and $\mathrm{emp}_{\hat{c}_{-i}}$.

Lemma 5 *Let $i \in I_n$ and $\eta > 0$. If $\max_{j \neq i} d(c_j, \hat{c}_j) \leq \eta$, then*

$$\rho(\mathrm{emp}_{c_{-i}}, \mathrm{emp}_{\hat{c}_{-i}}) \leq \eta.$$

Proof. Let D be a Borel measurable subset of $T \times X$. Since $c_j \in D$ implies that $\hat{c}_j \in \overline{B}_\eta(D)$, then

$$\mathrm{emp}_{c_{-i}}(D) = \frac{|\{j \neq i : c_j \in D\}|}{n-1} \leq \frac{|\{j \neq i : \hat{c}_j \in \overline{B}_\eta(D)\}|}{n-1} = \mathrm{emp}_{\hat{c}_{-i}}(\overline{B}_\eta(D)). \tag{38}$$

Thus, $\rho(\mathrm{emp}_{c_{-i}}, \mathrm{emp}_{\hat{c}_{-i}}) \leq \eta$. ■

In particular, Lemma 5 implies that the function $\mathrm{emp} : c_{-i} \mapsto \mathrm{emp}_{c_{-i}}$ is continuous, which, in turn, implies that the function $c \mapsto V_i(c_i, \mathrm{emp}_{c_{-i}})$ is continuous. Hence, if $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma^n$ is a strategy, $\breve{\sigma}_l \in \mathcal{M}(T \times X)$ is defined by equation (1) for all $l \in I_n$ and $\breve{\sigma}_{-i} = \breve{\sigma}_1 \times \cdots \breve{\sigma}_{i-1} \times \breve{\sigma}_{i+1} \cdots \times \breve{\sigma}_n$, we can define

$$U_i(\sigma) = \int_{C^n} V_i(c_i, \mathrm{emp}_{c_{-i}}) \mathrm{d}\breve{\sigma}(c) = \int_C \int_{C^{n-1}} V_i(c_i, \mathrm{emp}_{c_{-i}}) \mathrm{d}\breve{\sigma}_{-i}(c_{-i}) \mathrm{d}\breve{\sigma}_i(c). \tag{39}$$

For convenience, let

$$U_i(x, \sigma_{-i}|t) = \int_{C_{-i}} V_i(t, x, \mathrm{emp}_{c_{-i}}) \mathrm{d}\breve{\sigma}_{-i}(c_{-i}) \tag{40}$$

for all $i \in I_n$, $t \in T$, $x \in X$ and $\sigma \in \Sigma^n$.

The following lemma asserts that $(\mathcal{K}, K)$ can be approximated by $(\mathcal{H}, K)$ when both T and X are finite.

Lemma 6 *If both T and X are finite, then $(\mathcal{K}, K)$ can be approximated by $(\mathcal{H}, K)$.*

Proof. We may assume that $V_i(t, x, \mu) \in [0, 1]$ for all $i \in I_n$, $t \in T$, $x \in X$ and $\mu \in \mathcal{M}(T \times X)$. Let $\varepsilon > 0$, $\sigma \in \Sigma^n$ and $i \in I_n$. Let $0 < \eta < \min\{\varepsilon/2, 2\}$ and $\delta > 0$ be such that $|\mu - \nu| < \delta$ implies that $|V_i(t, x, \mu) - V_i(t, x, \nu)| < \eta$ for all $i \in I_n$.[2] Finally, let $N \in \mathbb{N}$ be such that $n \geq N$ implies that $4|C| \exp^{-2n\delta^2} < \eta$.

Since

$$|U_i(\sigma) - \hat{U}_i(\sigma)| \leq \sum_{c_i \in C} \breve{\sigma}_i(c_i) \left| \sum_{c_{-i} \in C^n} \breve{\sigma}_{-i}(c_{-i}) V_i(c_i, \mathrm{emp}_{c_{-i}}) - V_i(c_i, \bar{\sigma}_{-i}) \right|, \quad (41)$$

it is enough to show that

$$\left| \sum_{c_{-i} \in C^n} \breve{\sigma}_{-i}(c_{-i}) V_i(c_i, \mathrm{emp}_{c_{-i}}) - V_i(c_i, \bar{\sigma}_{-i}) \right| < 2\eta.$$

Note that if c_{-i} is such that $\mathrm{emp}_{c_{-i}} \in B_\delta(\bar{\sigma}_{-i})$, then

$$|V_i(c_i, \mathrm{emp}_{c_{-i}}) - V_i(c_i, \bar{\sigma}_{-i})| < \eta,$$

while if $\mathrm{emp}_{c_{-i}} \notin B_\delta(\bar{\sigma}_{-i})$, then

$$|V_i(c_i, \mathrm{emp}_{c_{-i}}) - V_i(c_i, \bar{\sigma}_{-i})| \leq 2.$$

Let $\gamma = \breve{\sigma}_{-i}(\{c_{-i} : \mathrm{emp}_{c_{-i}} \notin B_\delta(\bar{\sigma}_{-i})\})$. By Kalai (2004, Lemma 4), then $\gamma < 2|C| \exp^{-2n\delta^2}$ and so

$$\begin{aligned} &\left| \sum_{c_{-i} \in C^n} \breve{\sigma}_{-i}(c_{-i}) V_i(c_i, \mathrm{emp}_{c_{-i}}) - V_i(c_i, \bar{\sigma}_{-i}) \right| < 2\gamma + (1 - \gamma)\eta = \\ &= \eta + (2 - \eta)\gamma < \eta + 4|C| \exp^{-2n\delta^2} < 2\eta. \end{aligned} \quad (42)$$

[2]Since $T \times C$ is finite, we may think of μ and ν as a vector in $\mathbb{R}^{|T||X|}$. Then, $|\mu - \nu|$ denotes their Euclidian distance.

Hence, $|U_i(\sigma) - \hat{U}_i(\sigma)| < \varepsilon$. ■

It then follows from Lemma 6 and Corollary 2 that all Nash equilibria in sufficiently large games $G_n \in \mathcal{K}$ can be approximately purified.

Lemma 7 *Let K be an equicontinuous subset of $\mathcal{U}$ and $m, p \in \mathbb{N}$. Then, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if:*

1. *$n \geq N$,*
2. *$G_n \in \mathcal{K}$,*
3. *$V_n(I_n) \subseteq K$,*
4. *both T and X are finite, $|T| = p$ and $|X| = m$,*
5. *ν is a strong (ζ, η) – equilibrium of G_n for some $\zeta, \eta \geq 0$,*

then there exists an ε – purification g of ν such that $\rho(\bar{\nu}, \bar{g}) < \varepsilon$ and $E_n(\nu, \zeta) \subseteq E_n(g, \zeta + \varepsilon)$.

In order to generalize Lemma 7 to the general case of T and X compact, we proceed by approximation.

The following lemma estimates the difference in payoff that two strategies can create if they coincide in the rectangles of some γ – partitions of T and X. Before we state it, it is convenient to introduce the following notation. For all $V \in \mathcal{U}$ and $\delta > 0$, define $\omega(\delta) = \sup\{|V(t, x, \mu) - V(s, y, \nu)| : \max\{d(t, s), d(x, y), \rho(\mu, \nu)\} < \delta, t, s \in T, x, y \in X, \mu, \nu \in \mathcal{M}(T \times X)\}$.

Lemma 8 *Let $G_n \in \mathcal{K}$, $\delta > 0$, $\{(T_1, t_1), \ldots, (T_p, t_p)\}$ be a $\delta/4$ – partition of T and $\{(X_1, x_1), \ldots, (X_m, x_m)\}$ be a $\delta/4$ – partition of X.*

If σ and φ are strategies in G_n such that $\breve{\varphi}_i(T_l \times X_j) = \breve{\sigma}_i(T_l \times X_j)$ for all $i \in I_n$, $l = 1, \ldots, p$ and $j = 1, \ldots, m$, then

$$|U_i(x, \sigma_{-i}|t) - U_i(x, \varphi_{-i}|s)| < 2\omega_i(\delta)$$

for all $i \in I_n$, $x \in X$ and $t, s \in T$ such that $d(t, s) < \delta$.

Proof. The partitions $\{X_1, \ldots, X_m\}$ and $\{T_1, \ldots, T_p\}$ induce a partition $\{C_1, \ldots, C_{mp}\}$ of $T \times X$. Since $\breve{\varphi}_i(T_l \times X_j) = \breve{\sigma}_i(T_l \times X_j)$ for all $i \in I_n$, $l = 1, \ldots, p$ and $j = 1, \ldots, m$, then, $\breve{\varphi}_{-i}(\times_{j \neq i} C_{r_j}) = \breve{\sigma}_{-i}(\times_{j \neq i} C_{r_j})$ for all $i \in I_n$, $j \neq i$ and $r_j \in \{1, \ldots, mp\}$. Let $\Xi_{-i} = \{\times_{j \neq i} C_{r_j} : r_j \in \{1, \ldots, mp\} \text{ for all } j \neq i\}$.

Since

$$\begin{aligned} &\left| \int_{C_{-i}} V_i(s, x, \text{emp}_{c_{-i}}) \mathrm{d}\breve{\varphi}_{-i}(c_{-i}) - \int_{C_{-i}} V_i(t, x, \text{emp}_{c_{-i}}) \mathrm{d}\breve{\sigma}_{-i}(c_{-i}) \right| \\ &\leq \sum_{A \in \Xi_{-i}} \left| \int_A V_i(s, x, \text{emp}_{c_{-i}}) \mathrm{d}\breve{\varphi}_{-i}(c_{-i}) - \int_A V_i(t, x, \text{emp}_{c_{-i}}) \mathrm{d}\breve{\sigma}_{-i}(c_{-i}) \right|, \end{aligned} \tag{43}$$

it is enough to show that

$$\left| \int_A V_i(s, x, \text{emp}_{c_{-i}}) \mathrm{d}\breve{\varphi}_{-i}(c_{-i}) - \int_A V_i(t, x, \text{emp}_{c_{-i}}) \mathrm{d}\breve{\sigma}_{-i}(c_{-i}) \right| < 2\omega_i(\delta) \breve{\sigma}_{-i}(A)$$

for all $A \in \Xi_{-i}$.

Let $A \in \Xi_{-i}$ and $\hat{c}_{-i} \in A$. Hence, if $c_{-i} \in A$, then $d(c_j, \hat{c}_j) < \delta/2$ for all

$j \neq i$ and so $\rho(\text{emp}_{c_{-i}}, \text{emp}_{\hat{c}_{-i}}) \leq \delta/2$. Since $\breve{\varphi}_{-i}(A) = \breve{\sigma}_{-i}(A)$, it follows that

$$\begin{aligned}
&\left|\int_A V_i(s, x, \text{emp}_{c_{-i}}) \mathrm{d}\breve{\varphi}_{-i}(c_{-i}) - \int_A V_i(t, x, \text{emp}_{c_{-i}}) \mathrm{d}\breve{\sigma}_{-i}(c_{-i})\right| \\
&\leq \left|\int_A V_i(s, x, \text{emp}_{c_{-i}}) \mathrm{d}\breve{\varphi}_{-i}(c_{-i}) - \int_A V_i(t, x, \text{emp}_{\hat{c}_{-i}}) \mathrm{d}\breve{\varphi}_{-i}(c_{-i})\right| \\
&+ \left|\int_A V_i(t, x, \text{emp}_{\hat{c}_{-i}}) \mathrm{d}\breve{\varphi}_{-i}(c_{-i}) - \int_A V_i(t, x, \text{emp}_{\hat{c}_{-i}}) \mathrm{d}\breve{\sigma}_{-i}(c_{-i})\right| \\
&+ \left|\int_A V_i(t, x, \text{emp}_{\hat{c}_{-i}}) \mathrm{d}\breve{\sigma}_{-i}(c_{-i}) - \int_A V_i(t, x, \text{emp}_{c_{-i}}) \mathrm{d}\breve{\sigma}_{-i}(c_{-i})\right| \\
&< \omega_i(\delta)\breve{\varphi}_{-i}(A) + |\breve{\varphi}_{-i}(A) - \breve{\sigma}_{-i}(A)||V_i(t, x, \text{emp}_{\hat{c}_{-i}})| + \omega_i(\delta)\breve{\sigma}_{-i}(A) \\
&= 2\omega_i(\delta)\breve{\sigma}_{-i}(A).
\end{aligned} \tag{44}$$

Thus, the lemma follows. ■

The following lemma is the analog of Lemma 4 for the class $\mathcal{K}$.

Lemma 9 *Let K be an equicontinuous subset of $\mathcal{U}$ and $\eta > 0$. Then, there exist $m, p \in \mathbb{N}$ such that the following holds:*

For all games $G_n = (I_n, V_n, \tau, T, X) \in \mathcal{K}$ satisfying $V_n(I_n) \subseteq K$, there exists a game $\hat{G}_n = (I_n, \hat{V}_n, \hat{\tau}, \hat{T}, \hat{X}) \in \mathcal{K}$ such that

1. *both $\hat{T}$ and $\hat{X}$ are finite, $|\hat{T}| = p$ and $|\hat{X}| = m$;*
2. *$\hat{V}_i^n = V_i^n|_{\hat{T}\times\hat{X}\times\mathcal{M}(\hat{T}\times\hat{X})}$ for all $i \in I_n$;*
3. *If σ is a Nash equilibrium of G_n, then there exists a strong η – equilibrium ν of $\hat{G}_n$ such that $|U_i^n(\sigma) - U_i^n(\nu)| < \eta$; and*
4. *If g is a pure strategy strong ζ – equilibrium of $\hat{G}_n$ for some $\zeta \geq 0$, then there exists a pure strategy strong $\zeta + \eta$ – equilibrium f of G_n such that $|U_i^n(f) - U_i^n(g)| < \eta$.*

As in Section 4, we obtain the purification theorem for games in $\mathcal{K}$ with compact type and action spaces using the corresponding result for games with finite such spaces (Lemma 7) and an approximation lemma (Lemma 9).

Theorem 2 *Let K be an equicontinuous subset of $\mathcal{U}$. Then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with the following property:*

If $n \geq N$ and $G_n = (I_n, U_n, T, X) \in \mathcal{K}$ is such that $U_n(I_n)$ is a subset of K, then all Nash equilibria of G_n can be ε – purified.

5.2 Schmeidler (1973)

In this section, we consider games with a continuum of players as in Schmeidler (1973). Throughout, we assume that X is countable and T is a singleton, i.e., there is complete information.

The set of players is the $[0,1]$ interval endowed with the Lebesgue measure λ on the Borel measurable subsets of $[0,1]$. It follows that, in this case, there is no difference between the average of all players and all but player i.

A strategy is a measurable function $\sigma : [0,1] \to \mathcal{M}(X)$. The average choice $\bar{\sigma} \in \mathcal{M}(X)$ is defined by letting

$$\bar{\sigma}(x) = \int_{[0,1]} \sigma_i(x) \mathrm{d}\lambda(i). \tag{45}$$

It can be shown that

$$\bar{\sigma} = \int_{[0,1]} \sigma(i) \mathrm{d}\lambda(i) = \int_{\mathcal{M}(X)} \iota \mathrm{d}\lambda \circ \sigma^{-1}, \tag{46}$$

where $\iota : \mathcal{M}(X) \to \mathcal{M}(X)$ denotes the identity and the integrals are understood in the Gel'fand sense (see Appendix A.2).

A *game with a continuum of players* is characterized by a measurable function $V : [0, 1] \to \mathcal{U}$. We represent such a game by $G = ([0, 1], V, X)$.

Let Σ denote the set of all strategies. For all strategies $\sigma \in \Sigma$, let

$$\begin{aligned} E(\sigma) = \{ & i \in [0,1] : V_i(x, \bar{\sigma}) \geq V_i(\hat{x}, \bar{\sigma}) \\ & \text{for all } x \in \text{supp}(\sigma_i), \text{ and } \hat{x} \in X \} . \end{aligned} \tag{47}$$

We then say that σ is a *Nash equilibrium of* G if $\lambda(E(\sigma)) = 1$.

Define

$$U_i(\sigma) = \int_X V_i(x, \bar{\sigma}) \mathrm{d}\sigma_i(x) \tag{48}$$

for all $i \in [0, 1]$ and $\sigma \in \Sigma$. Clearly, σ is a Nash equilibrium if and only if $U_i(\sigma) \geq U_i(\tilde{\sigma}_i, \sigma_{-i})$ for all $\tilde{\sigma}_i \in \mathcal{M}(X)$ and almost all $i \in [0, 1]$.

The following is our purification result for this class of games.

Theorem 3 *Let G be a game with a continuum of players with X countable. Then, for all Nash equilibria σ of G, there exists a pure Nash equilibrium f of G such that $U_i(f) = U_i(\sigma)$ for all $i \in [0, 1]$.*

This result generalizes Theorem 2 in Schmeidler (1973) simply because we allow X to be countable and not merely finite.

As in the previous results, we prove Theorem 3 by approximation. In this case, we approximate games with a continuum of players with games with a finite number of players. Such approximation is better done using equilibrium distributions than equilibrium strategies. This explains why we need X to be countable: essentially, we show that if σ is a Nash equilibrium of a game G with a continuum of players, then it is also an approximate equilibrium in sufficiently large finite games. In those finite games, we can approximately purify it, obtaining a sequence of pure strategies. Despite the fact that it is not possible to obtain a pure strategy in G from this sequence, we can use it to obtain an equilibrium distribution of G. If X is countable, then we can obtain a pure strategy Nash equilibrium from this distribution (see Carmona (2006, Theorem 3), restated here as Lemma 14).

We next introduce the notion of (approximate) equilibrium distributions. Let τ be a Borel probability measure on $\mathcal{U} \times \mathcal{M}(X)$ and denote by $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{M}(X)}$ the marginals of τ on $\mathcal{U}$ and $\mathcal{M}(X)$, respectively. Define for all $\varepsilon \geq 0$

$$\begin{aligned} B_\tau^\varepsilon = & \left\{ (v,\sigma) \in \mathcal{U} \times \mathcal{M}(X) : v\left(x, \int_{\mathcal{M}(X)} \iota \mathrm{d}\tau_{\mathcal{M}(X)}\right) \geq \right. \\ & \left. v\left(\hat{x}, \int_{\mathcal{M}(X)} \iota \mathrm{d}\tau_{\mathcal{M}(X)}\right) - \varepsilon \text{ for all } x \in \operatorname{supp}(\sigma) \text{ and } \hat{x} \in X \right\} \end{aligned}$$

The set B_τ consists of the pairs (v,σ) such that all points x in the support of σ are within ε of the maximum value for v when the distribution of actions is the marginal of τ on $\mathcal{M}(X)$. Note that B_τ^ε is closed, and so a Borel set; hence $\tau(B_\tau^\varepsilon)$ is well defined.

A Borel probability measure τ on $\mathcal{U} \times \mathcal{M}(X)$ is an *equilibrium distribution* for a game G with a continuum of players if $\tau_{\mathcal{U}} = \lambda \circ V^{-1}$ and $\tau(B_\tau) = 1$.

The notion of an equilibrium distribution can also be defined for finite games. It turns out that for the approximation results needed to prove Theorem 3, it is convenient to work with a modified notion of equilibrium. Let $G_n = (I_n, V_n, X)$ be a finite game in $\mathcal{H}$ and let π_n denote the uniform distribution on I_n, i.e., $\pi_n(i) = 1/n$ for all $i \in I_n$. For all $\zeta, \eta \geq 0$, a Borel probability measure τ on $\mathcal{U} \times \mathcal{M}(X)$ is a *pseudo* (ζ, η) *– equilibrium distribution* for G_n if $\tau_{\mathcal{U}} = \pi_n \circ V_n^{-1}$ and $\tau(B_\tau^\zeta) \geq 1 - \eta$.

The difference between equilibrium and pseudo-equilibrium distribution will be clear once we associate pseudo-equilibrium distributions with strategies with similar properties. Let $G_n = (I_n, V_n, X)$ be a finite game in $\mathcal{H}$, $\sigma \in \Sigma^n$ be a strategy and $\zeta, \eta \geq 0$. We say that σ is a *pseudo* (ζ, η) – *equilibrium* if

$$\frac{|\{i \in I_n : V_i(x, \bar{\sigma}) \geq V_i(\hat{x}, \bar{\sigma}) - \zeta \text{ for all } x \in \text{supp}(\sigma_i), x \in X\}|}{n} \geq 1-\eta. \quad (49)$$

It is easy to show that σ is a pseudo (ζ, η) – equilibrium if and only if $\tau = \pi_n \circ (V_n, \sigma)^{-1}$ on $\mathcal{U} \times \mathcal{M}(X)$ is a pseudo (ζ, η) – equilibrium distribution for G_n. Furthermore, it is clear what the difference between a Nash equilibrium and a pseudo-equilibrium is: in the latter, each player computes his payoff using the average of all players including himself. More importantly, he assumes that when he changes his strategy, this average will be unchanged.

Formally, the difference between an (approximate) equilibrium and an (approximate) pseudo-equilibrium is that instead of imposing conditions on $V_i(x, \bar{\sigma}_{-i})$, we now impose them on $V_i(x, \bar{\sigma})$. Since in an n – players game, $\rho(\bar{\sigma}, \bar{\sigma}_{-i}) \leq 1/n$ by Lemma 13 in the Appendix, we can state Theorem 1 for pseudo-equilibria (here specialized for the case $|T| = 1$).

Corollary 3 *Let K be an equicontinuous subset of $\mathcal{U}$. Then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with the following property:*

If $n \geq N$ and $G_n = (I_n, V_n, X) \in \mathcal{H}$ is such that $V_n(I_n)$ is a subset of K, then for all $\zeta, \eta \geq 0$ and all pseudo (ζ, η) – equilibrium σ of G_n, there exists a pure strategy pseudo $(\zeta + \varepsilon, \eta)$ – equilibrium satisfying $|U_i(f) - U_i(\sigma)| < \zeta + \varepsilon$ and $\rho(\bar{\sigma}, \bar{f}) < \varepsilon$.

We next turn to our approximation results. The following lemma asserts that we can approximate Nash equilibria of games with a continuum of players with approximate pseudo-equilibria of games with finitely many players.

Lemma 10 *Let G be a game with a continuum of players and σ be a Nash equilibrium of G. Then, there exists a sequence of games $\{G_{n_k}\}_{k=1}^{\infty}$ such that*

1. *$n_k \to \infty$,*

2. *$\pi_{n_k} \circ (V, \sigma)^{-1}$ converges to $\lambda \circ (V, \sigma)^{-1}$ and*

3. *$\sigma|_{I_{n_k}}$ is a pseudo $(1/k, 1/k)$ – equilibrium of G_{n_k}.*

The following lemma is used to draw conclusions for games with a continuum of players from properties of large finite games.

Lemma 11 *Let $G = ([0,1], V, X)$ be a game with a continuum of players, τ be a distribution on $\mathcal{U} \times X$ satisfying $\tau_{\mathcal{U}} = \lambda \circ V^{-1}$ and $\varepsilon \geq 0$. Suppose that $\{G_{n_k}\}_{k=1}^{\infty}$ is a sequence of games with a finite number of players and $\{f_{n_k}\}_{k=1}^{\infty}$ is a sequence of pure strategies satisfying:*

1. *$n_k \to \infty$,*

2. *f_{n_k} is a pseudo (ε_k, η_k) – equilibrium of G_{n_k},*

3. *$\varepsilon_k \geq \varepsilon$, $\varepsilon_k \to \varepsilon$,*

4. *$\eta_k \geq 0$, $\eta_k \to 0$ and*

5. *$\pi_{n_k} \circ (V_{n_k}, f_{n_k})^{-1}$ converges to τ,*

then τ is an ε – equilibrium distribution of G.

The proofs of both Lemma 10 and Lemma 11 are a straightforward modification of Carmona (2004a, Theorem 1), and are therefore omitted.

Finally, we turn to the proof of Theorem 3.

Proof of Theorem 3. Let σ be a Nash equilibrium of G. Then, by Lemma 10, there exists a sequence of games $\{G_{n_k}\}_{k=1}^{\infty}$ such that

1. $n_k \to \infty$,

2. $\pi_{n_k} \circ (V, \sigma)^{-1}$ converges to $\lambda \circ (V, \sigma)^{-1}$ and

3. $\sigma|_{I_{n_k}}$ is a pseudo $(1/k, 1/k)$ – equilibrium of G_{n_k}.

Let $k \in \mathbb{N}$ be fixed. Then $\{V_i^{n_k}\}_{i \in I_{n_k}}$ is a finite, hence equicontinuous subset of $\mathcal{U}$. For all $\gamma \in \mathbb{N}$, let $G_{\gamma n_k}$ be the γth replica of G_{n_k}. Clearly, the γth replica $\sigma_{\gamma n_k}$ of σ_{n_k} is a pseudo $(1/k, 1/k)$ – equilibrium of $G_{\gamma n_k}$.

Then, if γ_k is sufficiently large, by Corollary 3, there exists a pure pseudo $(2/k, 1/k)$ – equilibrium $f_{\gamma_k n_k}$ satisfying $\rho(\bar{\sigma}_{\gamma_k n_k}, \bar{f}_{\gamma_k n_k}) < 1/k$. Let $\tau_k = \pi_{\gamma_k n_k} \circ (V_{\gamma_k n_k}, f_{\gamma_k n_k})^{-1}$; τ_k is a probability measure on $\mathcal{U} \times \mathcal{M}(X)$. We may assume that $\gamma_k n_k > k$, by choosing γ_k large enough.

Since $\tau_{\mathcal{U},k} = \pi_{\gamma_k n_k} \circ V_{\gamma_k n_k}^{-1}$ converges to $\mu = \lambda \circ V^{-1}$, it follows that $\{\mu, \tau_{\mathcal{U},1}, \tau_{\mathcal{U},2}, \ldots\}$, and so $\{\tau_{\mathcal{U},k}\}_k$ is tight by Hildenbrand (1974, Theorem 32 and 33, p. 49 and 50). Also, since $\mathcal{M}(X)$ is compact, then $\{\tau_{X,1}, \tau_{X,2}, \ldots\}$ is tight by Hildenbrand (1974, Theorem 34, p. 50). Thus, $\{\tau_k\}_k$ is tight (Hildenbrand (1974, Theorem 35, p. 50)) and, taking a subsequence if necessary, we may assume that $\{\tau_k\}$ converges (Hildenbrand (1974, Theorem 31, p. 49)). Let $\tau = \lim_k \tau_k$. Then, by Lemma 11 it follows that τ is an equilibrium distribution of $\tau_{\mathcal{U}} = \mu$ satisfying $\tau(\mathcal{U} \times \{1_x : x \in X\}) = 1$. This last property follows since $\mathcal{U} \times \{1_x : x \in X\}$ is closed, and so $\tau(\mathcal{U} \times \{1_x : x \in X\}) \geq \lim_k \tau_k(\mathcal{U} \times \{1_x : x \in X\}) = 1$.

Since $\tau(\mathcal{U} \times \{1_x : x \in X\}) = 1$, restrict τ to $\mathcal{U} \times X$ by defining $\mu(B) = \tau(\{(v, 1_x) : (v, x) \in B\})$ for all Borel measurable $B \subseteq \mathcal{U} \times X$. Clearly, for all Borel measurable $D \subseteq X$,

$$\tau_{\mathcal{M}(X)}(\{1_x : x \in D\}) = \tau(\mathcal{U} \times \{1_x : x \in D\}) = \mu(\mathcal{U} \times D) = \mu_X(D). \quad (50)$$

By Lemma 18, it follows that $\int_{\mathcal{M}(X)} \iota \mathrm{d}\tau_{\mathcal{M}(X)} = \mu_X$.

Let

$$A_\mu = \{(v, x) \in \mathcal{U} \times X : v(x, \mu_X) \geq v(\hat{x}, \mu_X) \text{ for all } \hat{x} \in X\}. \quad (51)$$

Hence, $(v, x) \in A_\mu$ if and only if $v(x, \mu_X) \geq v(\hat{x}, \mu_X)$ for all $\hat{x} \in X$ if and only if $v(x, \int_{\mathcal{M}(X)} \iota \mathrm{d}\tau_{\mathcal{M}(X)}) \geq v(\hat{x}, \int_{\mathcal{M}(X)} \iota \mathrm{d}\tau_{\mathcal{M}(X)})$ for all $\hat{x} \in X$ if and only if $(v, 1_x) \in B_\tau$. Therefore, $\mu(A_\mu) = \tau(B_\tau) = 1$. By Lemma 14, it follows that there exists a function $\hat{f} : [0, 1] \to X$ such that $\mu_X = \lambda \circ \hat{f}^{-1}$ and $V_i(\hat{f}_i, \lambda \circ \hat{f}^{-1}) \geq V_i(x, \lambda \circ \hat{f}^{-1})$ for all $x \in X$.

Finally, define $f : [0, 1] \to \mathcal{M}(X)$ by $f(i) = 1_{\hat{f}(i)}$ for all $i \in [0, 1]$. Then,

$$\int_{[0,1]} f \mathrm{d}\lambda = \int_{\mathcal{M}(X)} \iota \mathrm{d}\lambda \circ f^{-1} = \mu_X = \lambda \circ \hat{f}^{-1}$$

by Lemma 18, $\mathrm{supp}(f_i) = \hat{f}_i$ and so

$$V_i(x, \int_{[0,1]} f \mathrm{d}\lambda) \geq V_i(\hat{x}, \int_{[0,1]} f \mathrm{d}\lambda)$$

for all $x \in \mathrm{supp}(f_i)$ and $\hat{x} \in X$. Hence, f is a Nash equilibrium of G.

We claim that $\int_{[0,1]} \sigma \mathrm{d}\lambda = \int_{[0,1]} f \mathrm{d}\lambda$. Since

$$\int_{[0,1]} \sigma \mathrm{d}\lambda = \int_{\mathcal{M}(X)} \iota \mathrm{d}\lambda \circ \sigma^{-1} = \lim_k \int_{\mathcal{M}(X)} \iota \mathrm{d}\pi_{n_k} \circ \sigma_{n_k}^{-1},$$

$$\int_{[0,1]} f \mathrm{d}\lambda = \int_{\mathcal{M}(X)} \iota \mathrm{d}\lambda \circ f^{-1} = \lim_k \int_{\mathcal{M}(X)} \iota \mathrm{d}\pi_{n_k} \circ f_{n_k}^{-1},$$

it is enough to show that

$$\lim_k \int_{\mathcal{M}(X)} \iota \mathrm{d}\pi_{n_k} \circ \sigma_{n_k}^{-1} = \lim_k \int_{\mathcal{M}(X)} \iota \mathrm{d}\pi_{n_k} \circ f_{n_k}^{-1}.$$

This follows since $\rho(\int_{\mathcal{M}(X)} \iota \mathrm{d}\pi_{n_k} \circ \sigma_{n_k}^{-1}, \int_{\mathcal{M}(X)} \iota \mathrm{d}\nu_{n_k} \circ f_{n_k}^{-1}) = \rho(\bar{\sigma}_{n_k}, \bar{f}_{n_k}) < 1/k$.

Since f is a Nash equilibrium, then $U_i(f) = V_i(\hat{f}_i, \int f) = V_i(\hat{f}_i, \int \sigma) \geq V_i(x, \int \sigma)$ for all $i \in [0,1]$ and $x \in X$. Hence,

$$U_i(f) \geq \int_X V_i(x, \int \sigma) \mathrm{d}\sigma_i(x) = U_i(\sigma).$$

Similarly, $U_i(\sigma) = \int_X V_i(x, \int \sigma) \mathrm{d}\sigma_i(x) \geq V_i(\hat{f}_i, \int \sigma) = V_i(\hat{f}_i, \int f) = U_i(f)$ for all $i \in [0,1]$. Hence, $U_i(f) = U_i(\sigma)$ for all $i \in [0,1]$. ■

5.3 Cartwright and Wooders (2005)

Cartwright and Wooders (2005) also present a purification result for the class of games we consider. Although their result is neither implied nor implies our main result, some comparison can be made. In fact, their result is weaker to the extent that both their type and action spaces are countable. However, their result is stronger in the following aspects: their type and action spaces are not required to be compact, players' types are not required to be independent, their continuity notion is weaker than ours and their purification yields payoffs that are close to the original for all possible types (and not just in expected value, as in our results).

Our approach is, essentially, topological: once we strengthen Rashid's purification result for the finite model (i.e., with a finite number of types and actions), we use compactness and continuity to extend it. In contrast, Cartwright and Wooders (2005) address the purification problem directly by approximating mixed strategies with pure strategies. Their approach is important since it provides tools to directly address the countable case, and yields a purification result which is quite strong along many important dimensions. However, it seems that this strength needs to be compensated somehow, there, by placing assumptions on the cardinality of A and T.

5.4 Incomplete Information Games with Diffused Information

Our purification results rely on the fact that every player has a nearly negligible influence on the average. As the case of a continuum of players makes clear, this negligible influence of each player can be understood as a consequence of an (almost) atomless measure space of players.

Similarly, the purification of mixed strategies can also be obtained with a possible small number of players when their probability measure over types is atomless (which can be interpreted as a form of diffused information). Indeed, such results were obtained by Dvoretzky, Wald, and Wolfowitz (1951a), Aumann, Katznelson, Radner, Rosenthal, and Weiss (1983), Milgrom and Weber (1985) and Balder (2002), among others.

Although we have not done it, it seems likely that the approach we have used to address purification in large games, can also be used in the case of games with almost diffused information. In fact, a result along these lines was obtained by Rashid (1985). The reason for such belief relies partly on the generality of the Shapley-Folkman Theorem, which is the main tool in both our approach and that of Rashid (1985).

Indeed, as Khan, Rath, and Sun (2006) have pointed out, many of the above results can be seen as a consequence of the Dvoretzky-Wald-Wolfowitz Theorem,[3] which, in turn, is an extension of Lyapunov's Theorem. Alternatively, as shown by Balder (2005), we can regard those results as a consequence of the Lyapunov's theorem for Young measures of Balder (2000), which is again a consequence of Lyapunov's Theorem.[4] Since this theorem can be obtained using the Shapley-Folkman as shown by Tardella (1990), it follows that the purification results for games with diffused information rely, at least indirectly, on the Shapley-Folkman Theorem.

6 Games based on the Distribution of Individual Choices

In this section we show that, in general, it is not possible to approximately purify all Nash equilibria of sufficiently large games in which payoffs depend on the distribution of choices. This is so even if T is a singleton.

The games in this section are similar to those defined in Section 2, except that each player's preferences depend on his own choice and on the distribution of mixed strategies chosen by the other players (and not just on the average).

[3]See Dvoretzky, Wald, and Wolfowitz (1950, Theorem 1) and Dvoretzky, Wald, and Wolfowitz (1951b, Theorem 4) for its proof.

[4]In fact, the Lyapunov's theorem for Young measures is a consequence of the Extended Lyapunov Theorem of Balder (2000), which is, as its name indicates, an extension of Lyapunov's Theorem.

We assume that T is a singleton and, therefore, omit t from the following discussion. A strategy $\sigma = (\sigma_1, \ldots, \sigma_n)$ can be thought of as a function $\sigma : I_n \to \mathcal{M}(X)$. As before, π_n denotes the uniform measure on I_n and, for all $i \in I_n$, let $\pi_{n,i}$ denote the uniform measure on $I_n \setminus \{i\}$.

Let $\sigma \in \Sigma^n$ be a strategy and $i \in I_n$. The distribution $\pi_{n,i} \circ \sigma^{-1}$ of mixed strategies chosen by the all players other than player i is the element of $\mathcal{M}(\mathcal{M}(X))$ defined by

$$\pi_{n,i} \circ \sigma^{-1}(B) = \frac{|\{l \in I_n \setminus \{i\} : \sigma_l \in B\}|}{n-1} \tag{52}$$

for all Borel measurable subsets B of $\mathcal{M}(X)$. The quantity $\pi_{n,i} \circ \sigma^{-1}(B)$ is the fraction of players other than player i that choose a mixed strategy in B.

To each player $i \in I_n$, we associate a continuous function $V_i^n : X \times \mathcal{M}(\mathcal{M}(X)) \to \mathbb{R}$ with the following interpretation: $V_i^n(x, \mu)$ is player i's payoff when he plays action x and faces the distribution μ. Then, for any strategy σ, player i's payoff function is

$$U_i^n(\sigma) = \int_X V_i^n(x, \pi_{n,i} \circ \sigma^{-1}) \mathrm{d}\sigma_i(x). \tag{53}$$

This class of games is denoted by $\mathcal{D}$.

We claim that there exists $\varepsilon > 0$ and an equicontinuous subset K of the space of all continuous, real-valued functions on $X \times \mathcal{M}(\mathcal{M}(X))$ such that the following holds: For all $N \in \mathbb{N}$, there exists $n \geq N$, a game $G_n \in \mathcal{D}$ and a Nash equilibrium σ of G_n with no ε – purification.

Let $\varepsilon = 1/4$ and $X = \{a, b\}$. Then, $\mathcal{M}(X)$ can be represented by unit interval $[0, 1]$ with the convention that $y \in [0, 1]$ corresponds to the probability measure μ satisfying $\mu(\{a\}) = y$. For all $z \in [0, 1]$, let 1_z denote the probability measure on $[0, 1]$ degenerate on z. Let $B^c_{1/4}(1_{1/2})$ denote the complement of the open ball of radius 1/4 around $1_{1/2}$ and define $g : \mathcal{M}([0,1]) \to \mathbb{R}$ as follows:

$$g(\mu) = \frac{\rho\left(\mu, B^c_{1/4}(1_{1/2})\right)}{\rho\left(\mu, B^c_{1/4}(1_{1/2})\right) + \rho\left(\mu, 1_{1/2}\right)}. \tag{54}$$

Note that g is continuous, $g(1_{1/2}) = 1$ and $g(\mu) = 0$ for all $\mu \notin B_{1/4}(1_{1/2})$. Let $W : X \times \mathcal{M}([0,1]) \to \mathbb{R}$ be defined by $W(x,\mu) = g(\mu)$ for all $x \in X$. Let $K = \{W\}$.

Let $N \in \mathbb{N}$ and let $n = N$. Define G_n by letting $V^n_i = W$ for all $i \in I_n$. In particular, $V_n(I_n) = K$.

Consider σ defined by $\sigma_i(\{a\}) = 1/2$ for all $i \in I_n$. Then $\pi_{n,i} \circ \sigma^{-1} = 1_{1/2}$ and so $U^n_i(\sigma) = 1 \geq U^n_i(\varphi)$ for all players $i \in I_n$ and all strategies $\varphi \in \Sigma^n$. Hence, σ is a Nash equilibrium of G_n. We claim that there is no $\varepsilon -$ purification of σ. In fact, all pure strategies f satisfy $\rho(\pi_{n,i} \circ f^{-1}, 1_{1/2}) \geq 1/4$ for all $i \in I_n$, from which the conclusion follows since then $U^n_i(f) = 0$ and so

$$|U^n_i(\sigma) - U^n_i(f)| = 1 > \frac{1}{4} \tag{55}$$

for all $i \in I_n$.

So, it remains to prove that if f is a pure strategy, then $\rho(\pi_{n,i} \circ f^{-1}, 1_{1/2}) \geq 1/4$ for all $i \in I_n$. For convenience, denote $\pi_{n,i} \circ f^{-1}$ by τ. Let $D = \{1/2\}$. Then obviously $1_{1/2}(D) = 1$, $\overline{B}_{1/4}(D) = [1/4, 3/4]$ and $\tau(\overline{B}_{1/4}(D)) = 0$, since $\tau(\{0,1\}) = 1$. It follows that

$$1_{1/2}(D) = 1 > \frac{1}{4} = \tau(\overline{B}_{1/4}(D)) + \frac{1}{4}, \tag{56}$$

and so $\rho(\tau, 1_{1/2}) \geq 1/4$.

A Appendix

A.1 Lemmata

In this appendix, we prove several results needed for our main results. Lemma 12 deals with measures with a finite support. For such measures, we will sometimes write μ_l instead of $\mu(\{l\})$, whenever l is a point in the space in which μ is strictly positive. This notation also suggests that a measure with a finite support can be thought of as a vector in some Euclidean space. Roughly, Lemma 12 says that the Prohorov distance between two measures whose support is contained in some finite set is proportional to their Euclidean distance.

Lemma 12 *Let $\mu, \nu \in \mathcal{M}(Y)$ be such that* $\text{supp}(\mu) \cup \text{supp}(\nu) \subseteq \Psi$*, where Ψ is a finite set. If there exists $\varepsilon > 0$ such that $||\tau_l - \mu_l|| \leq \varepsilon$ for all $1 \leq l \leq |\Psi|$, then $\rho(\tau, \mu) \leq |\Psi|\varepsilon$.*

Proof. Let $\varepsilon > 0$ and $B \subseteq Y$ be Borel measurable. Then,

$$\begin{aligned} \mu(B) = \sum_{l \in \Psi \cap B} \mu(\{l\}) \leq \sum_{l \in \Psi \cap B} (\nu(\{l\}) + \varepsilon) \leq \\ \leq \sum_{l \in \Psi \cap B} \nu(\{l\}) + |\Psi|\varepsilon \leq \nu(\overline{B}_{|\Psi|\varepsilon}(B)) + |\Psi|\varepsilon. \end{aligned} \tag{57}$$

Similarly, we can show that $\nu(B) \leq \mu(\overline{B}_{|\Psi|\varepsilon}(B)) + |\Psi|\varepsilon$. This implies that $\rho(\mu, \nu) \leq |\Psi|\varepsilon$. ■

The following lemma shows that, in large games, the impact of one player on the average distribution of type-action characters is small.

Lemma 13 *Let σ be a strategy in a game G_n and $i \in I_n$. Then,*

$$\rho(\bar{\sigma}, \bar{\sigma}_{-i}) \leq \frac{1}{n}.$$

Proof. Let D be a Borel measurable subset of $T \times X$. Note that

$$\begin{aligned} \bar{\sigma}(D) = \frac{1}{n}\sum_{l \in I_n} \sigma_l(D) &= \frac{n-1}{n}\frac{1}{n-1}\sum_{l \neq i} \sigma_l(D) + \frac{1}{n}\sigma_i(D) \\ &= \frac{n-1}{n}\bar{\sigma}_{-i}(D) + \frac{1}{n}\sigma_i(D). \end{aligned} \tag{58}$$

Hence, one easily sees that both $\bar{\sigma}(D) \leq \bar{\sigma}_{-i}(D) + 1/n$ and $\bar{\sigma}_{-i}(D) \leq \bar{\sigma}(D) + 1/n$ hold. Thus, $\rho(\bar{\sigma}, \bar{\sigma}_{-i}) \leq 1/n$. ■

The following result, proved in Carmona (2006), shows that we can obtain a pure strategy Nash equilibrium from an equilibrium distribution when the action space is countable.

Lemma 14 *Let G be a game with a continuum of players with X countable. If τ is an equilibrium distribution of G, then there exists a Nash equilibrium $f : [0, 1] \to X$ such that $\tau_X = \lambda \circ f^{-1}$.*

We conclude this subsection with a lemma used to approximate games with a continuum of players with games with finitely many players. This lemma is a simple extension of Parthasarathy (1967, Theorem II.6.3) and is proven in Carmona (2004a).

Lemma 15 *Let $(T, \mathcal{T}, \lambda)$ be an atomless measure space, X a separable metric space, $\mu \in \mathcal{M}(X)$ and $K \subseteq \operatorname{supp}(\mu)$ be compact. If $\mu = \lambda \circ h^{-1}$, where $h : T \to X$ is measurable, then there exists a sequence $\{\mu_n\}$ in $\mathcal{M}(X)$ such that*

1. $\mathrm{supp}(\mu_n) \subseteq \mathrm{supp}(\mu)$ *for all* $n \in \mathbb{N}$*,*

2. $\mu_n \Rightarrow \mu$*,*

3. $\lim_n \mu_n(K) = \mu(K)$

4. *for all* $n \in \mathbb{N}$*,* $\mu_n = \nu_n \circ h^{-1}_{|T_n}$ *where* T_n *is a finite subset of* T *and* ν_n *is the uniform measure on* T_n *and*

5. $|T_n| \to \infty$*.*

A.2 Gel'fand Integration

Let $ca(X)$ denote the AL-space of all signed measures with bounded variation on the Borel subsets of X and $C_b(X)$ be the space of all bounded, continuous real-valued functions on X. Then, the pair $\langle C_b(X), ca(X)\rangle$ is a dual pair under the duality $W(f,\mu) = \langle f, \mu\rangle = \int_X f \mathrm{d}\mu$ (see Aliprantis and Border (1999), especially footnote 1 on page 475).

Let $(\Omega, \Upsilon, \varphi)$ be a measure space. A function $g : \Omega \to \mathcal{M}(X)$ is *weak* measurable* if for all $h \in C_b(X)$, the function $\omega \mapsto W(h, g(\omega))$ is measurable. Note that if $G : C_b(X) \times \Omega \to C_b(X) \times \mathcal{M}(X)$ is defined by $G(h,\omega) = (h, g(\omega))$, then that function is just $W \circ G$. Hence, if g is measurable, then g is weak* measurable. In particular, all strategies $\sigma : [0,1] \to \mathcal{M}(X)$ are weak* measurable.

Let g be weak* measurable. Then, μ is the *Gel'fand integral of* g if for all $h \in C_b(X)$,

$$\int_X h \mathrm{d}\mu = \int_\Omega \left(\int_X h \mathrm{d}g(\omega) \right) \mathrm{d}\varphi(\omega).$$

As usual, one writes $\mu = \int_\omega g\mathrm{d}\varphi$. Note that, by Aliprantis and Border (1999, Theorem 11.51), $\int_\omega g\mathrm{d}\varphi$ is well defined for all weak* measurable functions g if $\varphi(\Omega)$ is finite. In particular, all strategies are Gel'fand integrable.

The following lemma characterizes the Gel'fand integral of a strategy when the action space is countable.

Lemma 16 *Let X be countable and $\sigma : [0,1] \to \mathcal{M}(X)$ be a strategy. If*

$$\mu(x) = \int_{[0,1]} \sigma_i(x)\mathrm{d}\lambda(i)$$

for all $x \in X$, then

$$\mu = \int_{[0,1]} \sigma\mathrm{d}\lambda.$$

Proof. Let $h \in C_b(X)$. Then,

$$\begin{aligned}\int_X h\mathrm{d}\mu &= \sum_{j=1}^{\infty} h(x_j)\mu(x_j) = \sum_{j=1}^{\infty} h(x_j)\int_{[0,1]} \sigma_i(x)\mathrm{d}\lambda(i) = \\ &= \sum_{j=1}^{\infty}\int_{[0,1]} h(x_j)\sigma_i(x)\mathrm{d}\lambda(i) = \int_{[0,1]}\sum_{j=1}^{\infty} h(x_j)\sigma_i(x_j)\mathrm{d}\lambda(i) = \\ &= \int_{[0,1]}\int_X h\mathrm{d}\sigma(i)\mathrm{d}\lambda(i).\end{aligned} \tag{59}$$

So, $\mu = \int_{[0,1]} \sigma\mathrm{d}\lambda$. ■

Lemma 16 together with the standard change of variable formula (see Hildenbrand (1974, Theorem 36, p. 50)) yield the follows result.

Lemma 17 *Let $g : \Omega \to \mathcal{M}(X)$ be measurable and $\alpha : \mathcal{M}(X) \to \mathcal{M}(X)$ be continuous. Then,*

$$\int_{\mathcal{M}(X)} \alpha\mathrm{d}\varphi \circ g^{-1} = \int_\Omega \alpha \circ g\mathrm{d}\varphi.$$

In particular, if $\alpha = \iota$ and $\sigma : [0,1] \to \mathcal{M}(X)$ is a strategy, then

$$\int_{\mathcal{M}(X)} \iota \mathrm{d}\lambda \circ \sigma^{-1} = \int_{[0,1]} \sigma \mathrm{d}\lambda.$$

The following result considers the case of distributions induced by pure strategies.

Lemma 18 *Let X be countable and τ be a Borel probability measure on $\mathcal{M}(X)$ satisfying $\tau(\{1_x : x \in X\}) = 1$. If $\mu \in \mathcal{M}(X)$ is such that*

$$\mu(B) = \tau(\{1_x : x \in B\})$$

for all Borel measurable $B \subseteq X$, then,

$$\int_{\mathcal{M}(X)} \iota \mathrm{d}\tau = \mu.$$

Proof. By definition, we have that $\mu = \int_{\mathcal{M}(X)} \iota \mathrm{d}\tau$, if

$$\int_X h \mathrm{d}\mu = \int_{\mathcal{M}(X)} \left(\int_X h \mathrm{d}\iota(\nu) \right) \mathrm{d}\tau(\nu), \tag{60}$$

for all $h \in C_b(X)$.

Let $h \in C_b(X)$. Then,

$$\int_X h \mathrm{d}\mu = \sum_{j=1}^{\infty} h(x_j)\mu(x_j) = \sum_{j=1}^{\infty} h(x_j)\tau(1_{x_j}). \tag{61}$$

Also, $\int_X h \mathrm{d}\iota(\nu) = \int_X h \mathrm{d}\nu$. If $\nu = 1_x$ for some $x \in X$, then $\int_X h \mathrm{d}\nu = h(x)$ and so

$$\int_{\mathcal{M}(X)} \left(\int_X h \mathrm{d}\iota(\nu) \right) \mathrm{d}\tau(\nu) = \int_{\{1_x : x \in X\}} \left(\int_X h \mathrm{d}\nu \right) \mathrm{d}\tau(\nu) = \sum_{j=1}^{\infty} h(x_j)\tau(1_{x_j}). \tag{62}$$

Therefore, $\mu = \int_{\mathcal{M}(X)} \iota \mathrm{d}\tau$. ■

Lemma 18 has the following application. If $f : [0,1] \to \mathcal{M}(X)$ is a pure strategy and if $\hat{f} : [0,1] \to X$ is defined by $\hat{f}(i) = x$ if $\mathrm{supp}(f(i)) = \{x\}$ for all $i \in [0,1]$, then

$$\int_{[0,1]} f \mathrm{d}\lambda = \lambda \circ \hat{f}^{-1},$$

i.e., the integral of a pure strategy viewed as a function from players into degenerate probability measures over X equals its distribution when it is viewed as a function from players into actions.

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